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Distance-preserving flag fault-tolerant protocols for planar color codes of distance 9

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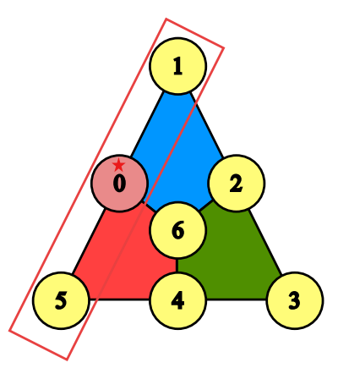
FLAG FAULT-TOLERANT ERROR CORRECTION

Quantum Error Correcting codes can handle the code capacity noise model – i.e. memory errors between rounds. However, syndrome measurements can be faulty in realistic settings.



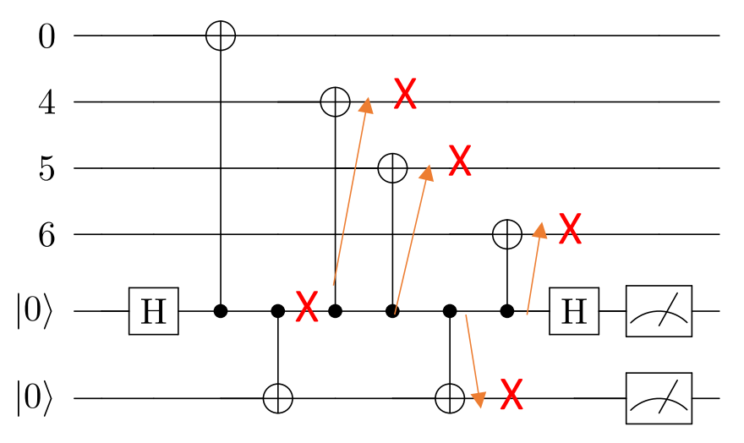
$$H = \begin{pmatrix} 1&0&0&0&1&1&1\\ 0&0&1&1&1&0&1\\ 1&1&1&0&0&0&1 \end{pmatrix} \qquad E = \begin{pmatrix} 1&0&0&0&0&0&0 \end{pmatrix} \quad \vec{s} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

In this work [1], we separate **space decoding** and efficient **time decoding.**

Time decoding can handle untrustworthy syndrome measurements by repetition, e.g. using Shor's method [2] or our special protocols [1].

$$\begin{pmatrix}1\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\1\\0\\1\\0\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\1\\0\\0\\0\\0\\0\end{pmatrix} \cdots \begin{pmatrix}1\\0\\1\\0\\0\\0\\0\\1\end{pmatrix} \xrightarrow{\text{time decoder}} \begin{pmatrix}1\\0\\0\\0\\1\\1\\0\\1\end{pmatrix} \xrightarrow{\text{space decoder}} R = X_2Y_3\ldots$$

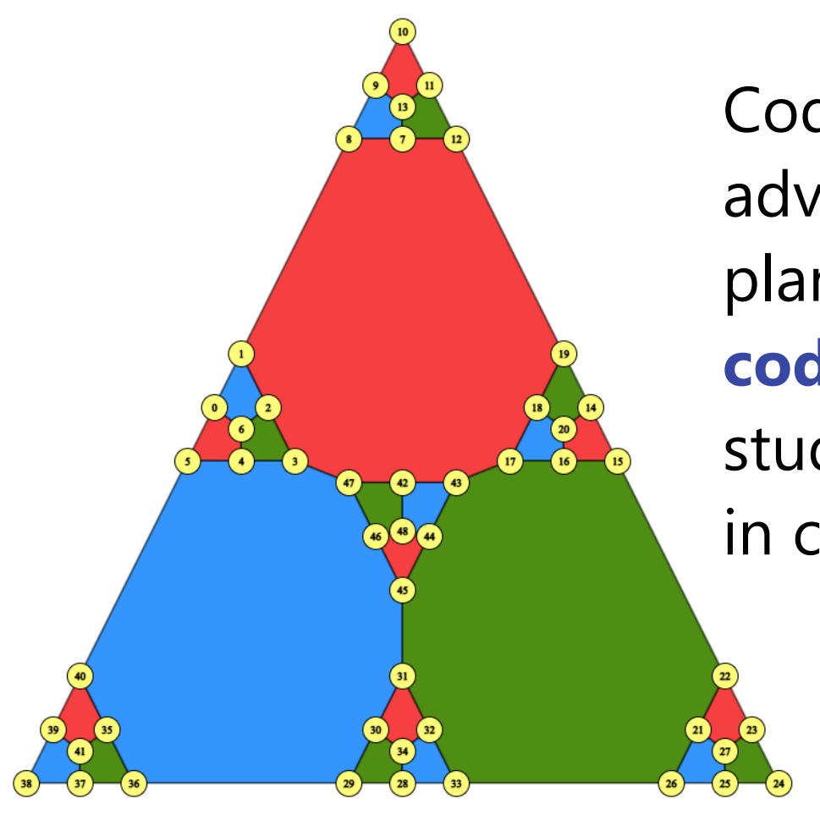
Faults on ancillas can propagate back to the data qubits as high-weight errors. **Flag qubits** [3] can **distinguish** these concerning errors from low-weight ones even if they have the same syndrome with the right **space decoder.**

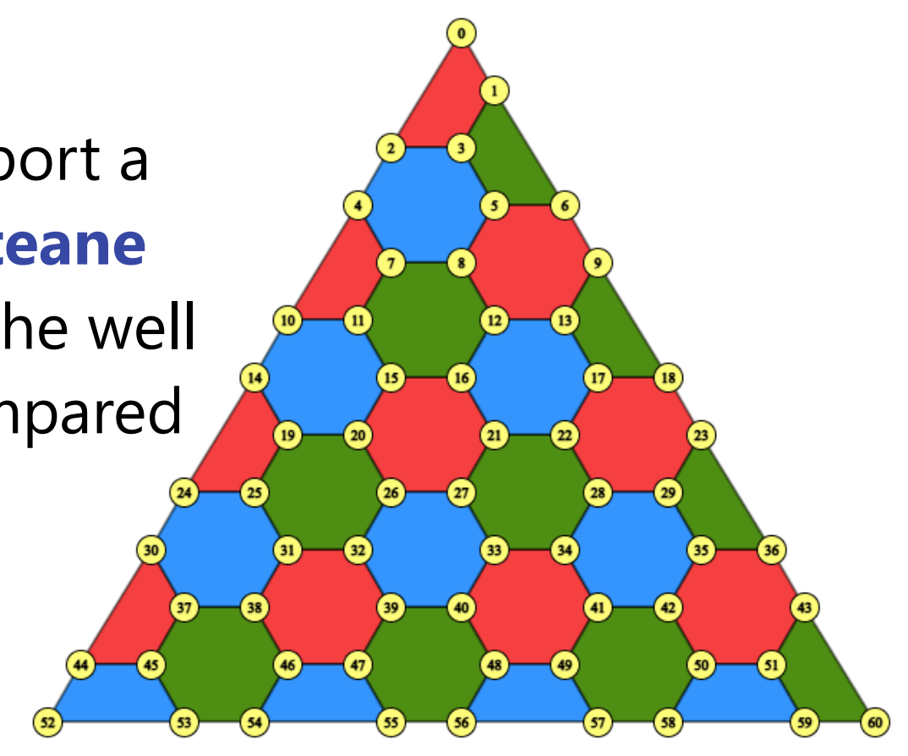


To be able to distinguish the accepted syndrome $\vec{s}(E)$ of the combined data error *E*, we concatenate it with the cumulative flag vector $\vec{f} = \sum \vec{f}^{(i)}$ - the combined object is the **full syndrome**, which is used by the **space decoder** to generate the recovery operator.

PLANAR DISTANCE 9 COLOR CODES

Codes that have a **planar layout** can be advantageous for certain architectures. We report a planar layout for the **Level 2 concatenated Steane code** [1] and we compare its performance to the well studied **triangular color code** (previously compared in code capacity error model in [4]).


[[49,1,9]] Level 2 Steane code

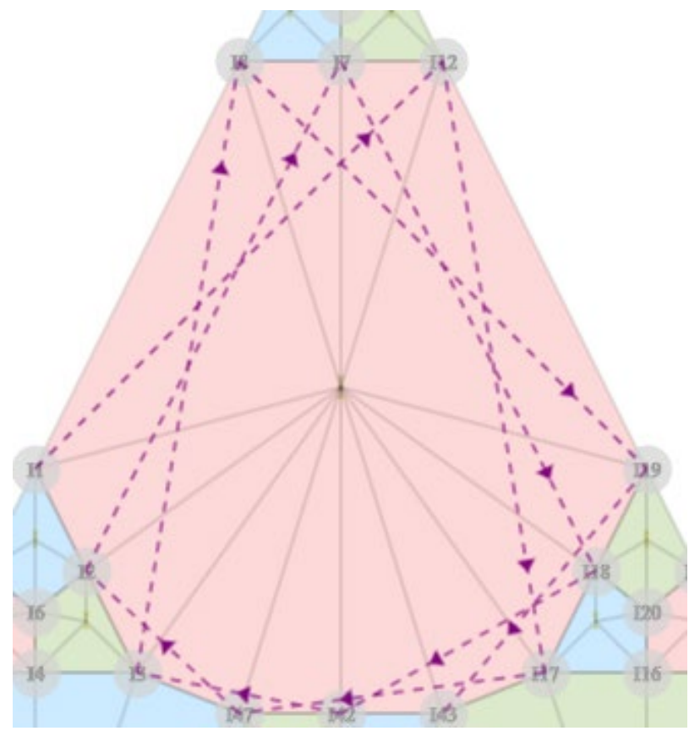

[[61,1,9]] triangular color code

DISTINGUISHABILITY

Definition 1 (following [5]): Let the fault set $\mathcal{F}_t$ denote the set of all possible fault combinations arising from up to *t* faults and let *S* be the stabilizer group of the underlying QEC code. We say that $\mathcal{F}_t$ is **distinguishable** if for any pair of fault combination Λ_p, Λ_q in $\mathcal{F}_t$,

$$\vec{s}(E_p) \neq \vec{S}(E_q), \text{ or } \vec{f_p} \neq \vec{f_q}, \text{ or } E_pE_q \in S$$

Under a noise model that has 1 and 2 qubit depolarizing channel after 1 and 2 qubit gates, using only 1 flag qubit and a 1 ancilla qubit (per stabilizer generator if needed) and Shor's repeated measurement protocol **we found a CNOT schedule for the [[49, 1, 9]] code** that yields distinguishable full syndromes for each possible fault combination up to 4 faults – thus the protocol preserves the code distance.



NOISE MODEL

- **Depolarizing error model** parametrized by a single physical error parameter *p*
  - after each 1-qubit (2-qubit) gate 1-qubit (2-qubit) depolarizing channel of strength *p*, which leaves the state unchanged with probability *1-p*, and with probability p/3 (p/15) applies one of the possible 1-qubit (2-qubit) Pauli operators $\mathrm{X, Z, Y}(\{I, X, Z, Y\}^{\otimes 2} \backslash \{I \otimes I\})$.
  - With probability $\frac{2p}{3}$ flip the result of a **measurement** operation and *after* a **reset** operation flip the state to the orthogonal state
- Note, that exploring and combating the effect of idling noise is in progress and will be part of the preprint [1]. Distinguishability is not impacted by lack of idling errors.

SPACE DECODING AND DISTANCE VERIFICATION

We use a general lookup table-based distance verification and decoder for an [[n, k, d]] stabilizer code under depolarizing noise models.

Ingredient 1: Compact lookup table: maps a syndrome for Pauli operators of weight up to *t* to its **logical class**. All Pauli operator $P \in \mathcal{P}_n$ can be decomposed as pure errors (*E*), a stabilizer (*S*) and a logical operator $L \in \mathcal{P}_k$, $P = ESL$. We fix *E* to be the Paulis that flip exactly one syndrome bit (the right inverse of *H)*. The logical class is the parity of the Pauli frame after the application of the recovery operators on each single qubit error. We build all possible combinations up to *t* faults of the columns of the *logicalized parity check matrix* $\bar{H}|_{H^{-1}} = \begin{pmatrix} H \\ L \end{pmatrix}$ and verify that they are all unique (~ calculating the spark of the matrix). E.g., for the [[7,1,3]] code:

$$H = \begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1 \end{pmatrix}, \; H^{-1} = \begin{pmatrix} 0&0&1\\ 0&1&0\\ 0&0&0\\ 1&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix} \qquad L = J(\overbrace{H^{-1}H \oplus I_7}^{\text{recoveries}}) \qquad \begin{pmatrix} H \\ L \\ W \end{pmatrix} = \left(\begin{array}{ccccccc} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1\\ \hline 0&0&1&0&1&1&0\\ \hline 1&1&1&1&1&1&1 \end{array}\right)$$

J=(11…1) for parity

1 qubit errors

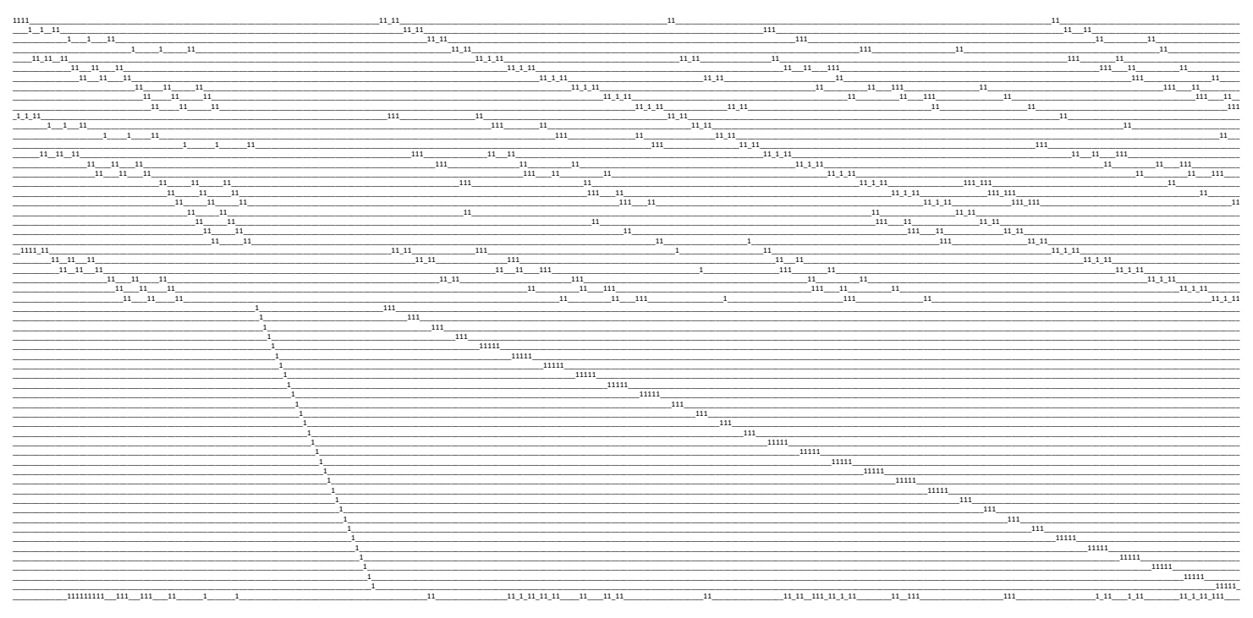
Ingredient 2: The effective code maps a fault combination to a full syndrome analogously to how the **error correcting code** maps a combination of single-qubit errors to generator bits.

↓ pure data faults ↓ Syndrome ancilla faults

$$\bar{H}_{\text{eff}} = \left(\begin{array}{c|c|c} H & 0 & \\ 0 & I_r & \bar{H}_{\text{flagged}} \\ J(H^{-1}H \oplus I_n) & 0 & \end{array} \right)$$

← Generator bits
← Flag bits
← Logical class

↑ Flag ancilla faults



Ingredient 3: Meet-in-the-middle optimization to increase the chance of decoding fault combinations with syndromes that are not in the lookup table.

TIME DECODER OPTIMIZATIONS

We optimize **adaptive syndrome measurements** [6] to achieve increased pseudo-thresholds. For a sequence of full syndromes (syndrome history) of length *n*, the **difference vector** $\vec{\delta}$ is a binary string of length *n-1*, with $\vec{\delta_i} = 0$ if $\vec{s}_i = \vec{s}_{i+1}$, and 1 otherwise. Suppose η_i are zero substrings of length $\gamma_i$and v' is the number of flag bits above 1 in flag syndromes between $\dot{|}$ and $\ddot{|}$, and

$$\vec{\delta_i} = \underbrace{\eta_1 1 \ldots 1\eta_{i-1}}_{\substack{\alpha \\ \mu}} \dot{|} 1\eta_i 1 \ddot{|} \underbrace{\eta_{i+1} 1 \ldots 1\eta_c}_{\substack{\beta \\ \omega}}, \text{ with}$$

min. # of faults
# of flag bits

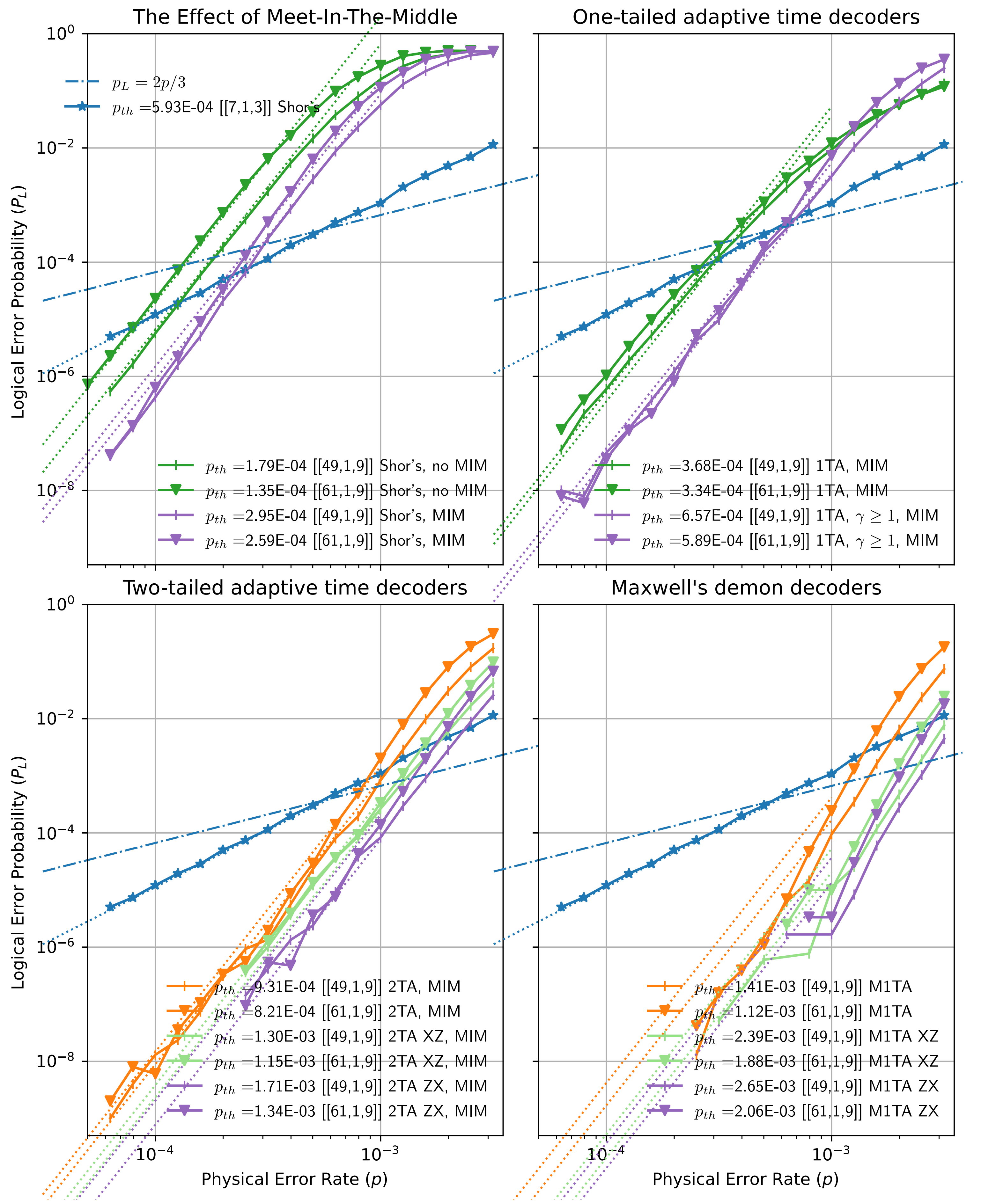
The **Fault Count Assumption (FCA)**: there are at most ***t*** faults in the syndrome history. The FCA fails with probability p^{t+1}. Using the FCA, these stopping conditions ensure with probability p^{t+1} that the accepted zero substring is trustworthy:

- **Shor's:** stop when $\gamma_c = t$, accept η_c, trade-off: none
- **1TA: One-tailed adaptive: a.)** $\gamma_c \geq 0$, stop when $\alpha + \gamma_c \geq t$, accept η_c **b.)** $\gamma_c \geq 1$, trade-off: none
- **2TA: Two-tailed adaptive:** stop when $\max(\alpha, \mu) + \gamma_i + v' + \max(\beta, \omega) \geq t$, accept η_i **-** trade-off: not obvious how to use the left-over flag information in fault-tolerant computation, only in storage
- **Two-tailed separate XZ/ZX:** 2TA on X-syndromes, estimate t_x, then 2TA on Z-syndromes with fault count target $t - t_x$ (for **ZX** swap X and Z) - trade-off: doesn't scale well for larger codes

For numerical upper bounds, we use **Maxwell's demon decoder.** Suppose the omniscient **Maxwell's demon** tells us the exact number of faults per round:

- **M1TA:** stop when last round had 0 faults and accept that as the syndrome
- **M1TA XZ/ZX: M1TA** on X-syndromes, then **M1TA** on Z-syndromes

NUMERICAL RESULTS



Space decoder	Time decoder	[[49,1,9]]	[[61,1,9]]	Maxwell [[49,1,9]]	Maxell [[61,1,9]]
No MIM	Shor's	1.79E-04	1.35E-04		
MIM	Shor's	2.95E-04	2.59E-04		
MIM	1TA	3.68E-04	3.34E-04		
MIM	1TA, γ≥1	6.57E-04	5.89E-04	1.41E-03	1.12E-03
MIM	2TA	9.31E-04	8.21E-04		
MIM	2TA XZ	1.30E-03	1.15E-03	2.39E-03	1.88E-03
MIM	2TA ZX	1.71E-03	1.34E-03	2.65E-03	2.06E-03

Methods: Direct sampling using Cirq[7], Stim/Sinter[8], Python/C++, Slurm. Number of samples per datapoint vary from a a minimum of 10^5 to a maximum of 10^9. Code and data will be open sourced as part of [1].

Conclusions: Both space and time decoding optimizations can have a significant effect. [[49, 1,9]] code slightly outperforms the [[61,1,9]] near threshold. We are not saturating the upper bounds but our time decoders are roughly within the same order of magnitude.

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